Lattice Flavourdynamics
Status and Prospects

Chris Sachrajda

School of Physics and Astronomy
University of Southampton
Southampton
UK

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Introduction

- In this talk I will discuss some of the key issues in lattice computations in flavour physics.
  - Chiral Extrapolations;
  - Heavy Quark Actions;
  - Kaon Physics.
- My main goal is to facilitate the interpretation of lattice results by the wider particle physics community.

  Statements of the form "Lattice gives XXX" hide a variety of inputs which significantly effect the systematic uncertainties.
- This talk is deliberately **not** a compilation of results.

  The status of lattice results for specific CKM-UT related quantities has been reviewed in the working groups.

Thanks for helpful correspondence to S.Aoki, C.Davies, A.Kronfeld, T.Onogi, G.Schierholz, S.Sharpe & M.Wingate

Lattice Flavourdynamics - Mission Statement

Lattice Simulations of QCD, in partnership with Experiments and Theory, play a central rôle in

- the determination of the fundamental parameters of the Standard Model (e.g. CKM matrix elements, quark masses);
- in searches of signatures of *New Physics*
- and potentially in understanding the structure of the new physics.

The principal reason for performing lattice simulations in Flavour Physics is, of course, to quantify non-perturbative QCD effects.

There is a very large community of theorists working in lattice QCD, improving our understanding of the systematics, developing new theoretical and computational techniques to reduce the uncertainties and extending the range of quantities which can be computed in lattice simulations.

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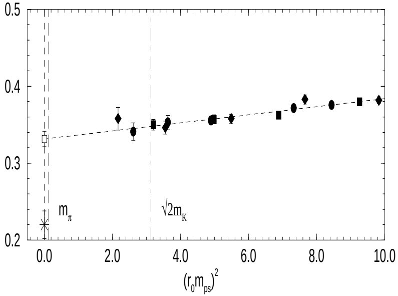
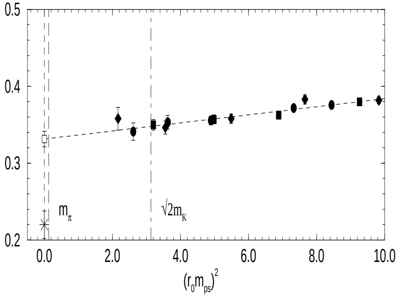
- It is not always the results which are presented the most loudly which are the most trustworthy.



Salutary Warning



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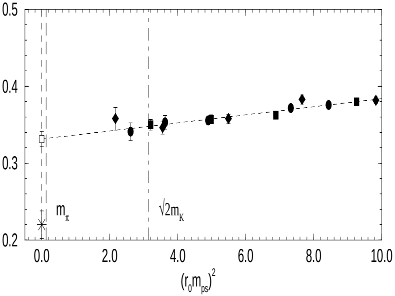
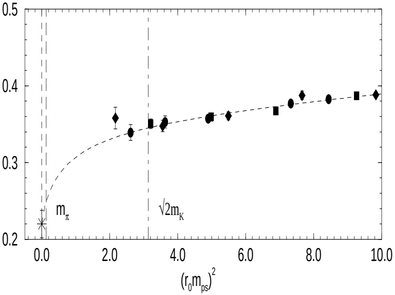
M.Göckeler et al.
(QCDSF Collaboration)
Phys.Rev.D **71** (2005) 114511





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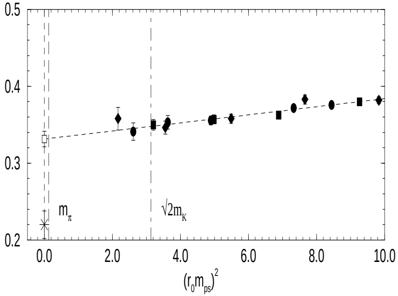

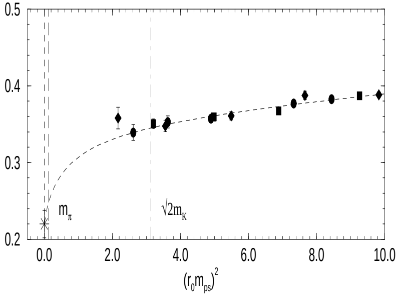

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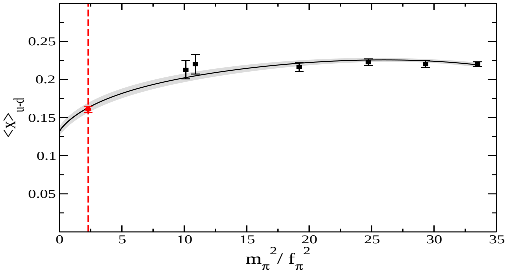
- These are quenched results.

Similar Features are present in *Unquenched* Simulations



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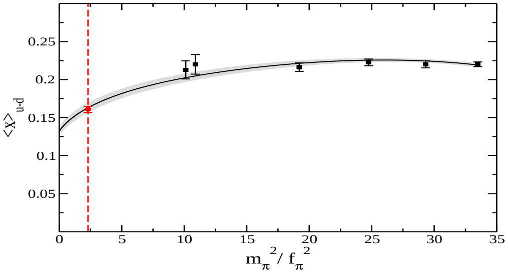

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- These results come from a *mixed action* simulation, with domain wall valence quarks on an improved staggered sea.
- This example illustrates the importance of reducing uncertainties due to the *chiral extrapolation*.

Chiral Extrapolations

- The emphasis now is on reducing the masses of the u and d quarks in the simulations, so as to control the chiral extrapolation.

	m_q/m_s	m_π (MeV)	m_π/m_ρ
$SU(3)$ Limit	1	690	0.68
Currently Typical	1/2	490	0.55
Impressive	1/4	340	0.42
MILC	1/8	240	0.31
Physical	1/25	140	0.18

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- The challenge has been set by the MILC collaboration (and collaborating groups using their data) using *Improved Staggered Fermions*, who have calculated many quantities with small quoted errors.

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High-Precision Lattice QCD Confronts Experiment

Rooted Staggered Fermions

- Staggered fermions $\Rightarrow$ 4 *tastes* of quark for each flavour.
- In practice this problem is tackled by *rooting*:

$$\int [dU] e^{-S_{\text{gauge}}} \det[D(m_u)] \det[D(m_d)] \det[D(m_s)] \quad \rightarrow$$

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- Steve Sharpe reviewed this question at Lattice 2006 asking whether this procedure was:
  - Good – correct continuum limit without any complications;
  - Bad – wrong continuum limit;
  - Ugly – correct continuum limit, but with unphysical contributions present for $a \neq 0$ requiring theoretical understanding and complicated fits.

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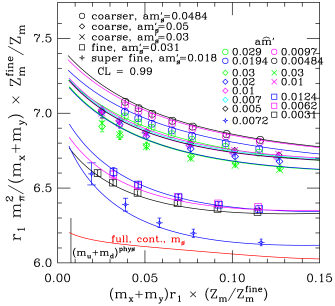
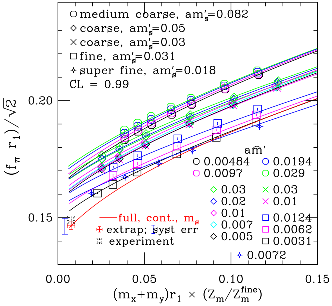
Are Rooted Staggered Fermions Bad or Ugly?

- Steve Sharpe, based on theoretical arguments using perturbation theory, the renormalization group and χPT (as well as on the circumstantial evidence from many numerical results), argues that it is *plausible that the correct choice is ugly*, i.e. that rooted staggered fermions are unphysical for $a \neq 0$, yet the results go over to those of a single-taste theory in the continuum limit. S.Sharpe, [hep-lat/0610094]
- Physical results must be extracted using complicated forms of rSχPT $\Rightarrow$ unphysical effects included in the fit.
- In any case we need to check the results against those obtained with other actions.
- The challenge set by the MILC Collaboration is being taken up by other collaborations using different lattice actions.





- After performing the *complicated fit* (978 data points with 28+26=54 parameters – including the analytic NNLO terms), the MILC results are extremely impressive.
- A selection of results:

$$f_\pi = 128.6 \pm 0.4 \pm 3.0\,\mathrm{MeV}; \qquad f_K = 155.3 \pm 0.4 \pm 3.1\,\mathrm{MeV};$$

$$m_s^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 90(0)(5)(4)\,\mathrm{MeV} \qquad \hat{m}^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 3.3(0)(2)(2)\,\mathrm{MeV}$$

C.Bernard et al. (MILC Collaboration) [hep-lat/0609053]

- The CERN Group using the DD-HMC algorithm on Wilson fermions find that the cost of generating 100 independent configurations with $N_f = 2$ on $2L \times L^3$ lattices in the ranges:

$$\frac{1}{4} \le \frac{\hat{m}}{m_s} \le 1; \qquad 0.052\,\text{fm} \le a \le 0.078\,\text{fm}; \qquad 1.2\,\text{fm} \le L \le 2.5\,\text{fm}$$

  is well described by the empirical formula

$$N_{\text{op}} = 0.05 \left(\frac{20\,\text{MeV}}{\hat{m}(2\,\text{GeV})}\right) \left(\frac{L}{3\,\text{fm}}\right)^5 \left(\frac{0.1\text{fm}}{a}\right)^6 \text{Tflops} \times \text{year}\,.$$

  L.Del Debbio et al., [hep-lat/0610059]

  Compared to *Berlin-Wall* formula (Lattice 2001) A.Ukawa, [hep-lat/0110179]

  - Scaling exponent of quark mass reduced from 3 to 1!
  - Exponent of lattice spacing reduced from 7 to 6;
  - Prefactor smaller by about 100.

- At Lattice 2006 a number of other collaborations also presented their results approaching $\sim$300 MeV for the pion mass with a variety of lattice actions.

  See for example Göckeler et al. [hep-lat/0610066], 320 MeV pions with Improved Wilson Action.

B-Physics

Among the recent results for f_{B_s}, the leptonic decay constant of the B-meson are:

$$f_{B_s} = 259(32)\,\mathrm{MeV}$$

A.Gray et al. (HPQCD Collaboration)
[hep-lat/0507015]

$$f_{B_s} = 191(6)\,\mathrm{MeV}$$

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NRQCD (non-relativistic QCD) + MILC Light Configurations

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HQET matched onto QCD + (Quenched) Improved Wilson

B-Physics

- $m_b \sim 4-5\,$GeV whilst typical lattice spacings are $a \sim 0.1\,$fm ($a^{-1} \sim 2\,$GeV) $\Rightarrow$ we cannot simulate the propagation of b-quarks directly in QCD.
- In early simulations results were obtained by extrapolating those obtained at lower masses ($m_Q \leq m_c$).
- Now the standard approach is to use effective theories and in this section I will discuss lattice simulations using:
  - HQET;
  - *Fermilab/Tsukuba/RHQ* Actions;
  - NRQCD.
- I will focus here on heavy-light physics, rather than on the physics of quarkonia.

HQET

$$\mathscr{L} = \bar{h} D_4 \left(\frac{1+\gamma^4}{2} \right) h \,.$$

The HQET action contains the non-perturbative QCD effects and can be matched onto QCD using perturbation theory, or better still non-perturbatively.

- For illustration, imagine that we compute the correlation function

$$C(t) = \sum_{\vec{x}} \langle 0 | A_4(\vec{x},t) A_4(\vec{0},0) \, | \, 0 \rangle$$

in the HQET. A_μ is the axial current, $A_\mu = \bar{h} \gamma_\mu \gamma^5 q$. h and q are heavy and light-quark fields.

- At large t:

$$C(t) \simeq Z^2 \exp(-\xi t)$$

where from Z we obtain the value of the decay constant, f_B, in the static approximation. E.Eichten (1987)

From the measured value of ξ one can obtain m_b (up to $\Lambda^2_{\rm QCD}/m_b$ corrections.) M.Crisafulli, V,Giménez, G.Martinelli, CTS [hep-ph/9506210]

m_b from HQET – Perturbative Matching

$$M_B = m_b^{\text{pole}} + \xi - \delta m$$

- ξ is computed numerically. It diverges linearly as $a \to 0$.
- δm is the perturbation series for the generated HQET mass

$$\delta m = \frac{1}{a} \sum_n X_n \, \alpha_s^n(m_b) \ .$$

  The linear divergence (partially) cancels that in ξ.
- The pole mass is not a physical quantity, containing a renormalon ambiguity. M.Beneke and V.Braun [hep-ph/9402364]; I.Bigi,M.Shifman, N.Uraltsev and A.Vainshtein [hep-ph/9402360].
  The renormalons in the pole mass and δm cancel.
- Thus from the measure value of ξ, together with perturbative calculations, we can obtain $m_b^{\overline{\text{MS}}}$.

m_b from HQET – Perturbative Matching (Cont.)

$$\begin{aligned}\overline{m}_b &= [1+\sum_n D_n\,\alpha_s^n(m_b)]\,m_B^{\rm pole} \\ &= [1+\sum_n D_n\,\alpha_s^n(m_b)][M_B-\xi+\frac{1}{a}\sum_n X_n\,\alpha_s^n(m_b)]\ .\end{aligned}$$

- The Parma group (and collaborators) has been developing stochastic perturbation theory to evaluate high-order perturbative coefficients.
  F. Di Renzo, V. Miccio, L. Scorzato and C. Torrero, [hep-lat/0609077]
  For δm they have calculated the three-loop coefficient.
  F. Di Renzo and L. Scorzato, [hep-lat/0409151]

$$\begin{array}{rl} \text{Quenched N}^3\text{LO} & \overline{m}_b = 4.30\pm0.05\pm0.05\,\text{GeV} \\ N_f=2\ \text{N}^3\text{LO} & \overline{m}_b = 4.21\pm0.03\pm0.05\pm0.04\,\text{GeV} \end{array}$$

V.Giménez, L.Giusti G.Martinelli and F.Rapuano (2000).

- The effect of the 3-loop term is to reduce the mass by about 50 MeV.

m_b from HQET – Non-Perturbative Matching

- Rainer Sommer and his collaborators at DESY-Zeuthen, Rome II and Berlin are pursuing a strategy to renormalize the HQET non-perturbatively. R.Sommer [hep-lat/0611020]
- The idea is based on *step-scaling*, which takes us from hadronic-size lattices of $O(\text{few fm})$ to very small lattices of $O(\text{fraction of a fm})$, where the lattice spacing is so small that QCD with a b quark can be simulated and perturbation theory is valid to excellent precision.
- I will present an oversimplified outline of the main steps, using m_b for illustration.
- As a sample observable consider an L^4 lattice:

$$\Gamma(L) \equiv \frac{\partial}{\partial t} \ln[-C(t)]_{t=L/2}\,.$$

On the hadronic-size lattice (L^4 say) $C(t) \sim Z^2 e^{-\xi t}$ and so $\Gamma[L] = \xi$.

m_b from HQET – Non-Perturbative Matching (Cont.)

- In going from QCD→ HQET we scale out a factor of e^{im_bt} (e^{-m_bt}) from the quark field:

$$M_B = m_b + \xi = m_b + \Gamma[L,a]$$

Since M_B is known and ξ is measured we know the numerical value of ξ (for the lattice HQET we are using).

- The aim of this procedure is to relate m_b non-perturbatively to some standard definition of the mass in QCD.

$$\begin{aligned} M_B &= m_b + (\Gamma[L,a] - \Gamma[L/2,a]) + \Gamma[L/2,a] \\ &= m_b + (\Gamma[L,a] - \Gamma[L/2,a]) + \Gamma[L/2,a/2] + (\delta m_b)_1 \end{aligned}$$

$$= m_b + (\Gamma[L,a] - \Gamma[L/2,a]) + (\Gamma[L/2,a/2] - \Gamma[L/2^2,a/2]) + \cdots + \Gamma[L/2^n,a/2^n] + (\delta m_b)_n\,.$$

- On the fine lattice we can impose:

$$\Gamma^{\rm QCD}(L/2^n,a/2^n) = \Gamma(L/2^n,a/2^n) + m_b + (\delta m_b)_n\,.$$

which (together with continuum perturbation theory) is an implicit equation for $\bar{m}_b$.

HQET with Non-Perturbative Matching

- Sommer et al. have recently extended this non-perturbative technique to include $O(\Lambda_{\rm QCD}/m_b)$ corrections:

$$\text{(Quenched)} \qquad \bar{m}_b = 4.347(48)\,\text{GeV}\,.$$

M. Della Morte, N. Garron, M. Papinutto and R. Sommer, [hep-ph/0609294]

- This technique now has to be extended to dynamical calculations and to other processes in b-physics.
- All other lattice computations in heavy quark physics rely on perturbative renormalization and matching.
- In general, when including $\Lambda_{\rm QCD}/m_b$ corrections, with perturbative matching, the first step is to make the uncertainties parametrically larger:

$$O\left(\frac{\Lambda_{\rm QCD}}{m_b}\right) \to O\left(\alpha_s^n(a)\,\frac{1}{m_b a}\right) = O\left(\frac{\alpha_s^n(a)}{a\Lambda_{\rm QCD}}\;\frac{\Lambda_{\rm QCD}}{m_b}\right).$$

In practice it is argued that α_s is sufficiently small that these induced errors are also small.

Relativistic Heavy Quark Action(s)

The aim is to construct lattice actions for simulations which will remove $O((m_Q a)^n)$ ($\forall n$) and $O(\Lambda_{\rm QCD} a)$ discretization errors.

- Start with lattice QCD and imagine adding all possible “irrelevant" terms necessary to *improve* the action a la Symanzik.
- Use the equations of motion to reduce the number of terms to the minimum required to achieve the required precision.

$$S = \sum_{n,n'} \bar\psi_{n'} \left(\gamma_4 D_4 + \zeta \vec{\gamma}\cdot\vec{D} + m_0 - \frac{r_t}{2} D_4^2 - \frac{r_s}{2}\vec{D}^2 + c_B \sum_{i,j} \frac{i}{4}\sigma_{ij}F_{ij} + c_E \sigma_{4i}F_{4i} \right)_{n'n} \psi_n .$$

- All higher dimensional operators which might be added to the action can be reduced to those above, up to terms of $O((\Lambda_{\rm QCD} a)^2)$.

This idea was first proposed by the Fermilab Group,

A.X.El-Khadra, A.S.Kronfeld & P.B.Mackenzie [hep-lat/9604004]

Relativistic Heavy Quark Action(s) - Cont

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How many of the 6 parameters above (ζ, m_0, $r_{t,s}$, $c_{E,B}$) need to be determined in order to ensure on-shell improvement?

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- Tsukuba - 5 — S.Aoki, Y.Kuramashi and S.Tominaga [hep-lat/0107009]

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- Columbia - 3 — N.H.Christ, M.Li & H-W.Lin [hep-lat/0608006]

Relativistic Heavy Quark Action(s) - Cont

$$S=\sum_{n,n'}\bar{\psi}_{n'}\left(\gamma_4 D_4+\zeta\vec{\gamma}\cdot\vec{D}+m_0-\frac{r_t}{2}D_4^2-\frac{r_s}{2}\vec{D}^2+c_B\sum_{i,j}\frac{i}{4}\sigma_{ij}F_{ij}+c_E\sigma_{4i}F_{4i}\right)_{n'n}\psi_n.$$

How many of the 6 parameters above (ζ, m_0, $r_{t,s}$, $c_{E,B}$) need to be determined in order to ensure on-shell improvement?

- Fermilab - 4 — A.X.El-Khadra, A.S.Kronfeld & P.B.Mackenzie [hep-lat/9604004]
- Tsukuba - 5 — S.Aoki, Y.Kuramashi and S.Tominaga [hep-lat/0107009]
- Columbia - 3 — N.H.Christ, M.Li & H-W.Lin [hep-lat/0608006]

(This is unfair; the different groups were not asking precisely the same question.)

Relativistic Heavy Quark Action(s) - Cont

$$S = \sum_{n,n'} \bar\psi_{n'} \left(\gamma_4 D_4 + \zeta \vec\gamma \cdot \vec D + m_0 - \frac{r_t}{2} D_4^2 - \frac{r_s}{2} \vec D^2 + c_B \sum_{i,j} \frac{i}{4} \sigma_{ij} F_{ij} + c_E \sigma_{4i} F_{4i} \right)_{n'n} \psi_n .$$

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(This is unfair; the different groups were not asking precisely the same question.)

Example of the reduction of the number of parameters:

$$\begin{aligned} \bar\psi \sigma_{4i} F_{4i} \psi &= \bar\psi \gamma_4 \gamma_i \left[D_4 D_i - D_i D_4 \right] \psi \\ &= -2 m_0\, \bar\psi \vec\gamma \cdot \vec D\, \psi - 4 \bar\psi \vec D^2\, \psi \end{aligned}$$

Thus a change in the coefficient c_E can be compensated by a change in the coefficients ζ and r_s .

Relativistic Heavy Quark Action(s) - Cont

$$S = \sum_{n,n'} \bar{\psi}_{n'} \left(\gamma_4 D_4 + \zeta \vec{\gamma}\cdot\vec{D} + m_0 - \frac{r_t}{2} D_4^2 - \frac{r_s}{2}\vec{D}^2 + c_B \sum_{i,j} \frac{i}{4} \sigma_{ij} F_{ij} + c_E \sigma_{4i} F_{4i} \right)_{n'n} \psi_n .$$

The Columbia group point out that a further parameter can be eliminated by making changes of variables in the fermion functional integral such as:

$$\psi \longrightarrow (1 + \chi \sigma_{4i}[D_i, D_4])\,\psi$$

They propose to set $r_s = r_t = 1$ and then tune the 3 parameters, m_0, ζ and $c_P \equiv c_E = c_B$.

N.Christ, M.Li & H-W.Lin [heo-lat/0608006] .

Lin and Christ also take the first step towards determining the parameters non-perturbatively. H-W Lin and N.Christ, [hep-lat/0608005]

Phenomenology with the Fermilab/Tsukuba Action

- The coefficients depend on the heavy quark mass.
- The phenomenological computations led by the Fermilab group have been performed with the improvement coefficients determined perturbatively at fixed a.
- For weak matrix elements, in addition to the action the operators must also be improved. Their coefficients are also obtained perturbatively.
- As a well-publicised example consider f_D, the leptonic decay constant of the D-meson.

$f_{D_s} = 249 \pm 3 \pm 16\,\text{MeV}$ $\quad$ $f_D = 201 \pm 3 \pm 17\,\text{MeV}$ $\quad$ C.Aubin et al. [hep-lat/050630]

The corresponding experimental results are

$$f_D = 223 \pm 17 \pm 3\,\text{MeV}$$ M.Artuso et al. (Cleo-c) [hep-ex/0508057]

$$f_{D_s} = 279 \pm 17 \pm 20\,\text{MeV}$$ J.W.Berryhill, BaBar (2006) (preliminary) [hep-ex/0508057]

Phenomenology with the Fermilab/Tsukuba Action Cont.

- Very recently the CP-PACS collaboration published a quenched numerical study of the Tsukuba actions. Y.Kayaba et al. [hep-lat/0611033]
  - $a^{-1} \simeq 2$ GeV;
  - Two gauge actions (Iwasaki and Plaquette);
  - Clover light-quark action;
  - Parameters in heavy quark action and in the currents determined at one-loop order in perturbation theory.
- The aim was to try to estimate the discretization effects by, for example, comparing f_D obtained with A_4 and with A_i (5% for Iwasaki and 0.5% for Plaquette Gauge Actions).

Relativistic Heavy Quark Actions - Concluding Remarks

The Fermilab approach and its generalizations are very interesting.

- It allows for the calculation of a variety of important quantities in heavy-quark physics (e.g. leptonic decay constants, form-factors for semileptonic decays, rare decay amplitudes).
- We need to add all possible improvement terms to the operators whose matrix elements we are computing with coefficients which have to be determined.
- In spite of the claims that the experimental measurements of f_D and $D \to K$ semileptonic decay form-factors are *precision tests of lattice QCD*, I will only feel comfortable when the coefficients are determined non-perturbatively.
- The coefficients of the neglected operators are functions of $m_Q a$ and one might worry that they become large, particularly for b-physics. However, the theory does have the correct static limit.

Non-Relativistic QCD (NRQCD)

- In the physics of heavy quarkonia the appropriate expansion parameter is the velocity ($p \sim v$ and $K \sim v^2$) and NRQCD is designed to facilitate this expansion.

NRQCD	Heavy Quark Expansion
$E = m(1 + O(v^2))$	$E = m + O(\Lambda_{\text{QCD}})$
$p = O(m * v)$	$p = O(\Lambda_{\text{QCD}})$
Expand in v.	Expand in Λ_{QCD}/m.

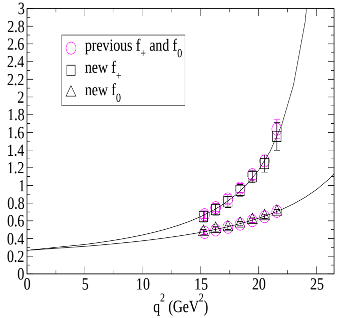
- NRQCD is also used in computations of quantities in Heavy-Light physics, where the HQET counting is relevant.

Non-Relativistic QCD (NRQCD)

- In practice the groups doing the calculations do their book-keeping and systematics differently:
- HQET
  - The kinetic term $\vec{D}^2/2m$ is treated as a perturbation.
  - The coefficients are being determined nonperturbatively (see above).
  - The continuum limit is taken.
- NRQCD
  - The kinetic term $\vec{D}^2/2m$ is treated as a full part of the action (and included in the propagator).
  - The coefficients are being determined perturbatively (estimates of higher order corrections are included in the errors).
    For example in the recent computation of the B-Meson semileptonic form-factors, the higher-order coefficients were all set to their tree-level values. E.Gulez et al. (HPQCD Collaboration) [hep-lat/0601021]
  - The calculations are performed at fixed a (ma is simply treated as a number).

NRQCD - Concluding Remarks

- For heavy-light physics, the Heavy Quark Expansion is appropriate.
- I believe that it is the differences in procedures described above, rather than the choice of action, which lead to the different results and capabilities.

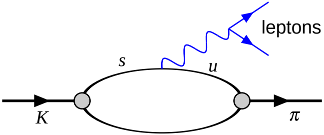


B-Meson Semileptonic Form Factors

(NRQCD + MILC Staggered Configurations)

E.Gulez et al. (HPQCD Collab.) [hep-lat/0601021]

$K_{\ell 3}$ Decays



$\Rightarrow \quad V_{us}$

$$\langle \pi(p_\pi)|\bar{s}\gamma_\mu l|K(p_K)\rangle = f^0(q^2)\,\frac{M_K^2-M_\pi^2}{q^2}q_\mu + f^+(q^2)\left[(p_\pi+p_K)_\mu - \frac{M_K^2-M_\pi^2}{q^2}q_\mu\right]$$

where $q \equiv p_K - p_\pi$ and $l = u,d$.

To be useful in extracting V_{us} we require $f^0(0) = f^+(0)$ to better than about 1% precision.

$$\chi\text{PT} \Rightarrow f^+(0) = 1 + f_2 + f_4 + \cdots \quad \text{where} \quad f_n = O(M^n_{K,\pi,\eta}).$$

Reference value $f^+(0) = 0.961 \pm 0.008$ where $f_2 = -0.023$ is relatively well known from χPT and $f_4, f_6, \cdots$ are obtained from models. Leutwyler & Roos (1984)

1% precision of $f^+(0)$ is conceivable because it is actually $1 - f^+(0)$ which is computed using *double ratios* such as: S.Hashimoto et al. [hep-ph/9906376]

$$\frac{\langle\pi|\bar{s}\gamma_4 l|K\rangle\langle K|\bar{l}\gamma_4 s|\pi\rangle}{\langle\pi|\bar{l}\gamma_4 l|\pi\rangle\langle K|\bar{s}\gamma_4 s|K\rangle} = \left[f_0(q^2_{\max})\right]^2 \frac{(m_K + m_\pi)^2}{4 m_K m_\pi}\,.$$

q^2 and m^2 extrapolations still have to be done.

2004 - Quenched Simulation $\Rightarrow f^+(0) =$ 0.960(9) . Bećirević et al. [hep-ph/0403217]
Unquenched (Improved Staggered) Result: $f^+(0) =$ 0.962(6)(9)
M.Okamoto (FNAL/MILC/HPQCD) [hep-lat/0412044]

2005 $N_f = 2$ results:

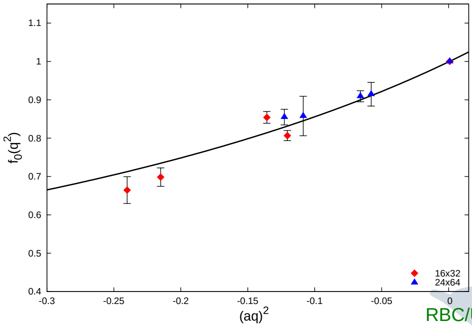
0.968(9)(6) C.Dawson et al. (RBC) [hep-ph/0607162]
0.952(6) N.Tsutsui et al. (JLQCD) [hep-lat/0510068]

$f^+(0) = 0.968(9)(6) \Rightarrow |V_{us}| = 0.2245(27)$ $(\text{PDG}(2006)\ K_{\ell 3} \Rightarrow |V_{us}| = 0.2257(21))$

2006 $N_f = 3$ DWF Result: $f^+(0) =$ 0.9680(16)
D.J.Antonio et al. (RBC/UKQCD) (preliminary) [hep-lat/0610080]

These results are in good agreement with the Leutwyler-Roos result of 0.961(8).

q^2 Dependence of the Form Factors



RBC/UKQCD (preliminary)

- For large $m_{u,d}$, $q^2_{\max} = (m_K - m_\pi)^2 \simeq 0$ and so the interpolation to $q^2 = 0$ is artificially small (in the plot $q^2_{\max} a^2 \simeq 8 \times 10^{-4}$)
- For the lightest quark mass in our simulation $q^2_{\max} a^2 \simeq .01$.
- For physical meson masses, $q^2_{\max} a^2 \simeq .05$.

(Partially) Twisted Boundary Conditions

- With periodic boundary conditions for the fields $\{\phi(x_i+L)=\phi(x_i)\}$, the components of momenta are quantized in units of $2\pi/L$.
- Typical Example:

$$L = 24a \quad \text{with} \quad a^{-1} = 2\,\text{GeV} \quad \Rightarrow \quad \frac{2\pi}{L} = .52\,\text{GeV}$$

$\Rightarrow$ available momenta for phenomenological studies are limited.

- Bedaque has advocated the use of twisted boundary conditions e.g. $q(x_i+L)=e^{i\theta_i}q(x_i)$, so that the momentum spectrum is

$$p_i = n_i\frac{2\pi}{L} + \frac{\theta_i}{L}\,.$$

P.Bedaque [nucl-th/0402051]

- For quantities which do not involve Final State Interactions the use of *partially twisted boundary conditions* in which the sea quarks satisfy periodic BC's but the valence quarks satisfy twisted BC's is valid (up to exponentially small corrections in the volume) $\Rightarrow$
  Do not need new simulations for every choice of $\{\theta_i\}$.

  CTS & G. Villadoro [hep-lat/0411023]; Bedaque & Chen [hep-lat/0412023]
- Can we calculate the form-factors directly at $q^2=0$?

$K_{\ell 3}$ Form-Factors Computed Directly at $q^2=0$ - Work in Progress

(UKQCD Collaboration) P.A.Boyle, J.Flynn, A.Jüttner, CTS & J.Zanotti (in progress)

- Consider the double ratio:

$$R^\mu(p_\pi,p_K)=\frac{\langle\pi(p_\pi)\,|\,V^\mu\,|\,K(p_K)\rangle\langle K(p_K)\,|\,V^\mu\,|\,\pi(p_\pi)\rangle}{\langle\pi(p_\pi)\,|\,V^\mu\,|\,\pi(p_\pi)\rangle\langle K(p_K)\,|\,V^\mu\,|\,K(p_K)\rangle}\,.$$

- With $\vec{p_\pi}=\vec{p_K}=0$ we get $f^0(q^2_{\rm max})$ to good precision:

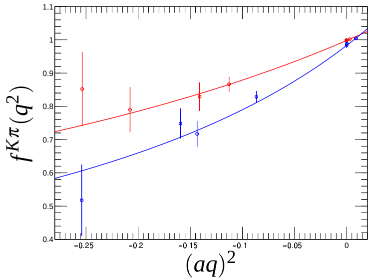
$$R^4(\vec{p}_\pi=0,\vec{p}_K=0)=\frac{(m_K+m_\pi)^2}{4m_Km_\pi}\left(f^0(q^2_{\rm max})\right)^2\quad\text{where}\quad q^2_{\rm max}=(m_K-m_\pi)^2.$$

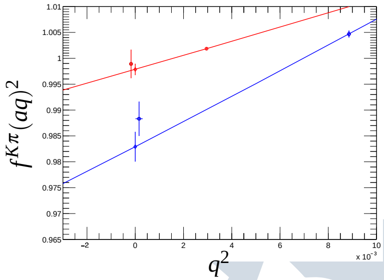
- Guadagnoli, Mescia, Simula have carried out an exploratory quenched study of the $K_{\ell 3}$ form factors using twisted boundary conditions.

  [hep-lat/0512020]

- If we set $\vec{p}_\pi=0$ ($\vec{p}_K=0$), then we can readily determine the values of $\vec{p}_K$ ($\vec{p}_\pi$) which give $q^2=0$. We can then use twisted boundary conditions to obtain the form factors at $q^2=0$ explicitly.
- In this way we can obtain $f^+(0)$ and avoid any uncertainty in the choice of ansatz in the interpolation in q^2.
- Can we do this with sufficient precision?

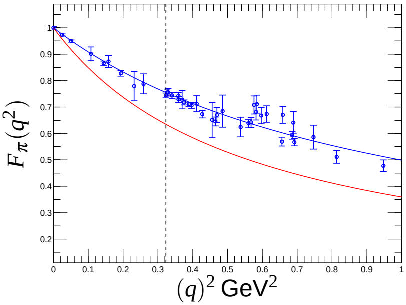
q^2-dependence of the form factor - Work in Progress





UKQCD+RBC Configs. $16^3 \times 32 \times 8$, DBW2+DWF, ma=0.02 (red) and 0.01 (blue)

Pion Form Factor - Work in Progress



- *Pion* Form Factor ($ma - 0.02$, $m_\pi \simeq 500$ MeV).
- With periodic boundary conditions we would be unable to compute points to the left of the dotted line.
- The size of the errors correlates well with the momenta of the mesons.

$K \to \pi\pi$ Decays - An Introduction

- A quantitative understanding of the non-perturbative QCD effects in $K \to \pi\pi$ decays is an important future milestone for lattice QCD, e.g.:
  - the empirical $\Delta I = 1/2$ rule, which states that amplitudes for decays with an $I = 0$ final state are enhanced by a factor of about 22 w.r.t. amplitudes for decays with an $I = 2$ final state.
  - the quantity ε'/ε, whose measurement with a non-zero value, $(17.2 \pm 1.8) \times 10^{-4}$, was the first observation of direct CP-violation.

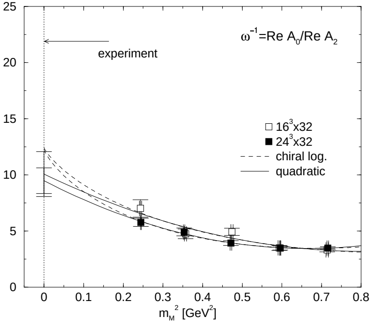
There continue to be two approaches to studying $K \to \pi\pi$ decays on a lattice:

- *Indirect Approach* in which $K \to \pi$ and $K -> 0$ matrix elements are computed and $K \to \pi\pi$ matrix elements are reconstructed using chiral perturbation theory. This was the approached used by the RBC & CP-PACS Collaborations in 2001:

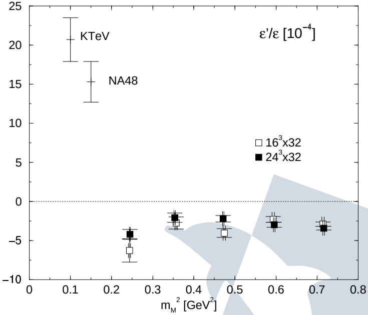
Collaboration(s)	Re A_4/Re A_2	ε'/ε
RBC*	25.3 ± 1.8	$-(4.0 \pm 2.3) \times 10^{-4}$
CP-PACS	9÷12	(-7 ÷ -2)$\times 10^{-4}$
Experiments	22.2	$(17.2 \pm 1.8) \times 10^{-4}$

*updated results from July 2002 version of the paper.

CP-PACS



ReA_0/**Re**A_2



ε'/ε

- Is χPT applicable/reliable in the accessible range (400-800 MeV)?
- Results from RBC and CP-PACS are very interesting and will provide valuable benchmarks for future calculations.
- One suggestion for the next stage is to improve the precision to NLO in the chiral expansion. Lin, Pallante, Martinelli, CTS & Villadoro
  Laiho & Soni
- In general we need to evaluate $K \to \pi\pi$ decay amplitudes directly.
- With two hadrons in the final state the finite-volume corrections decrease only as $1/L^n$ and not exponentially.
  The theory of finite-volume effects for two-hadron states in the elastic region is now fully understood $\Rightarrow$ $\pi\pi$ phase-shifts.

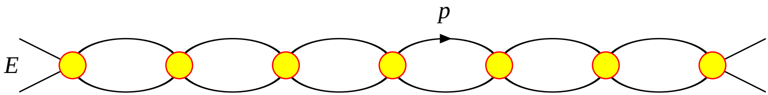
Finite-Volume Corrections for Two-Pion States

- M.Lüscher (1986-91) derived the two-hadron spectrum in a finite-volume in the rest frame.
- K.Rummukainen and S.Gottlieb (1996) generalized the derivation of the spectrum to a frame with non-zero momentum.
- L.Lellouch & M.Lüscher (2000) derived the finite-volume corrections to $K \to \pi\pi$ matrix elements in the rest frame.
- D.Lin, G.Martinelli, CTS & M.Testa (2001) rederived the spectrum (validating the results beyond the first 7 states) and the LL formula, interpreting the effects as being due to the density of two-pion states in a finite volume.
- C.Kim, CTS & S.Sharpe (2005) and N.Christ, C.Kim & T.Yamazaki (2005) generalized all the results to a moving frame.
- See also S.Beane, P.Bedaque, A.Parreno and M.Savage (2004).

In this talk I will simply outline why it is that the finite-volume corrections $\sim 1/L$ and do not decrease exponentially.

Finite-Volume Corrections for Two-Pion States

For two-particle states the finite-volume corrections decrease as powers of the volume and not exponentially. They are numerically significant and hence need to be controlled.



where $E^2 = 4(k^2 + m^2)$.

Performing the p_4 integration by contours we obtain summations over loop-momenta of the form:

$$\frac{1}{L^3} \sum_{\vec{p}} \frac{f(p^2)}{p^2 - k^2}$$

where $f(p^2)$ is non-singular.

For simplicity I am assuming here that only the s-wave $\pi\pi$ phase-shift is significant and that we are in the centre-of-mass frame. The generalization to higher partial waves is technical but straightforward.

$$\frac{1}{L^3}\sum_{\vec{p}}\frac{f(p^2)}{p^2-k^2}$$

- The *large* finite-volume effects, i.e. those which decrease only as powers of L, come from the difference between the momentum sums in finite-volume and the corresponding integrals in infinite volume.
- The required relation between the FV sums and infinite-volume integrals is the Poisson Summation Formula, which in 1-dimension is:

$$\frac{1}{L}\sum_{p} g(p) = \sum_{l=-\infty}^{\infty}\int \frac{dp}{2\pi} e^{ilLp} g(p)$$

If $g(p)$ is non-singular then only the term with $l=0$ on the rhs contributes, up to exponentially small terms in L.

- From the above it follows that this is not the case for two-hadron final states $\Rightarrow$ finite-volume corrections $\sim 1/L^n$.
- For two-hadron final states we start with

$$\frac{1}{L^3}\sum_{\vec{p}}\frac{f(p^2)-f(k^2)e^{\alpha(k^2-p^2)}}{p^2-k^2} = \int\frac{d^3p}{(2\pi)^3}\,\frac{f(p^2)-f(k^2)e^{\alpha(k^2-p^2)}}{p^2-k^2}.$$

This is the key formula to understanding FV effects in two-pion states.

Concluding Remarks – Chiral Extrapolations

Doubts as to the validity of using *rooted staggered fermions* have polarised the lattice community.

- Those who exploit the MILC configurations argue that there is considerable circumstantial evidence that the continuum limit, (which must be taken in a very *careful* way) is correct.
- The remainder of the community (numerically, the majority) argue that given the doubts, we cannot be sure that the results are correct.
- The advocates of different (local) formulations of lattice fermion actions are making very significant progress towards the chiral limit $\Rightarrow$ confidence in the results in the near future.
- (I find it a pity that the polarisation of the community means that the results obtained with improved staggered do not get checked (within the same formalism). Given the complicated and at best *ugly* (in the terminology of Steve Sharpe) formalism, this is regrettable. The same MILC configurations are used in most of the studies.)

Concluding Remarks – Heavy-Quark Actions

A variety of effective actions, in particular HQET, NRQCD and Fermilab/Tsukuba actions are being used in phenomenological applications.

- Rainer Sommer's group is making significant progress towards a controlled fully non-perturbative technique for computations in heavy quark physics. Continuum limit can be taken.
- Much heavy quark phenomenology is being done using NRQCD or the Fermilab/Tsukuba approach (often using the MILC configurations for the light quarks). The calculations are performed at fixed a, with the coefficients calculated in perturbation theory.
- The more one insists on theoretical rigour, the higher the price one has to pay in resources.

Concluding Remarks – Kaon Physics

- The evaluation of $f^+(0)$ for $K_{\ell 3}$ decays is possible at about the required precision.
- The use of (partially) twisted boundary conditions enables an improved momentum resolution for lattice phenomenology.
- Finite volume effects for the two-pion spectrum and $K \to \pi\pi$ amplitudes are understood in rest and moving frames.
  - For $I = 2$ final states, there is now no barrier to calculating the matrix elements precisely.
  - For $I = 0$ $\pi\pi$ states we need to learn how to calculate the disconnected diagrams with sufficient precision.